MTH 161 - Lecture 10

Lecture 10

THE PRODUCT AND QUOTIENT RULES

The product rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

Proof Let $F(x) = f(x)g(x)$

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0}\left[g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]$$

$$= \overbrace{\lim_{h\to 0} f(x+h)}^{\text{continuous}}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

In words, the product rule says :

Derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Example

- Differentiate $y = x^2 \sin x$

Soln Using Product rule,

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2)$$

$$= x^2 \cdot \cos x + \sin x \cdot (2x)$$

$$= x^2 \cdot \cos x + 2x \sin x$$

Example

- Differentiate $f(t) = \sqrt{t}\,(1 + 3t)$.

Method 1 (Product rule)

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(1+3t) + (1+3t)\frac{d}{dt}(\sqrt{t})$$

$$f'(t) = \sqrt{t}\,(3) + (1+3t)\,\frac{1}{2}t^{-1/2}$$

$$= 3\sqrt{t} + \frac{1}{2}t^{-1/2} + \frac{3}{2}t^{1/2}$$

$$= 3t^{1/2} + \frac{3}{2}t^{1/2} + \frac{1}{2}t^{-1/2}$$

$$= \frac{9t^{1/2}}{2} + \frac{1}{2}t^{-1/2}$$

Method 2

Rewrite $f(t) = t^{1/2} + 3t^{3/2}$

$$f'(t) = \frac{1}{2}t^{-1/2} + 3 \cdot \frac{3}{2}t^{1/2}$$

$$= \frac{1}{2}t^{-1/2} + \frac{9}{2}t^{1/2}$$

Same Answer. However the point is that it is not always easy to use the product rule. However Like in Ex 1, product rule is the only possible method.

Example

$h(x) = xg(x)$ and is let $g(3) = 12$ and $g'(3) = 16$, find $h'(3)$.

Soln

Use to product rule,

$$h'(x) = \frac{d}{dx}\left[xg(x)\right] = x\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}(x)$$

$$= xg'(x) + g(x)$$

Then, $h'(3) = 3 \cdot g'(3) + g(3)$

$$= 3 \cdot 16 + 12 = 48 + 12 = 60$$

THE QUOTIENT RULE

If f and g are differentiable functions, then,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{(g(x))^2}$$

The Quotient rule allows us to compute the derivative of rational functions.

lec 10

Let $y = \dfrac{x^2+x-2}{x^3+6}$.

Then,

$$y' = \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4+12x+x^3+6-3x^4-3x^3+6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

- Example Find an equation of the tangent line to the curve $y = \dfrac{\sqrt{x}}{1+x^2}$ at the point $\left(1, \frac{1}{2}\right)$.

Soln

To find the slope of the tangent line, we need to find the derivative

Then according to the Quotient rule,

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{\left((1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}(2x)\right)\cdot 2\sqrt{x}}{\left((1+x^2)^2\right)\cdot 2\sqrt{x}}$$

$$= \frac{(1+x^2) - 2\sqrt{x}\cdot\sqrt{x}\cdot(2x)}{2\sqrt{x}(1+x^2)^2} = \frac{1+x^2-4x^2}{2\sqrt{x}(1+x^2)^2} = \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$$

Then the slope of the tangent line at $(1, \frac{1}{2})$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1-3(1)^2}{2\sqrt{1}(1+1^2)^2} = \frac{1-3}{2\cdot 4} = \frac{-2}{8} = \frac{-1}{4}$$

Then using point slope form, the equation of the tangent line at $(1, \frac{1}{2})$

$$y - \frac{1}{2} = \frac{-1}{4}\left(x - \frac{1}{2}\right) \Rightarrow y = -\frac{1}{4}x + \frac{3}{4}$$

DON'T USE QUOTIENT RULE EVERY TIME YOU SEE A QUOTIENT

For instance,

$F(x) = \dfrac{3x^2+6x+8}{x}$, you can differentiate using the quotient rule. However it is much easier to perform division first and write $F(x) = 3x + 6 + 8x^{-1}$ and then differentiate.

TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x . \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of remaining trigonometric function $\csc x$, $\sec x$, $\cot x$ can be computed using quotient rule.

We obtain, $\frac{d}{dx}(\csc x) = -\csc x . \cot x$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Ex Let $f(\theta) = \dfrac{\sec\theta}{1+\sec\theta}$

Then, $f'(\theta) = \dfrac{(1+\sec\theta)\cdot(\sec\theta)' - \sec\theta\,(1+\sec\theta)'}{(1+\sec\theta)^2}$

$= \dfrac{(1+\sec\theta)\cdot(\sec\theta\tan\theta) - \sec\theta\,(\sec\theta\tan\theta)}{(1+\sec\theta)^2}$

$= \dfrac{\sec\theta\tan\theta + \sec^2\theta\cdot\tan\theta - \sec^2\theta\cdot\tan\theta}{(1+\sec\theta)^2}$

$= \dfrac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$

- Ex Differentiate $f(x) = \dfrac{x^3}{(1-x^2)}$. For what values of x does the graph of f have a horizontal tangent line?

Sol^n

$f'(x) = \dfrac{(1-x^2)\cdot(x^3)' - x^3(1-x^2)'}{(1-x^2)^2}$

$= \dfrac{(1-x^2)(3x^2) - x^3(-2x)}{(1-x^2)^2}$

$$= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2}$$

$$= \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2}$$

The horizontal tangent line mean slope of tangent line is 0, so we are looking for values of x such that $f'(x) = 0$.

In our case $x^2(3-x^2) = 0 \Rightarrow x = 0, x = \pm\sqrt{3}$ $\boxtimes$